Proof-by-Contradiction and Proof-by-Contraposition

Proof-by-Contradiction

To Prove: Statement p.

	Task:
Proof: (by Contradiction)	
Suppose ~p .	[Suppose ~p]
. . .	
∴ q	[∴ q]
. . .	
∴ ~q	[∴ ~q]
∴ q ∧ ~q (“This is a contradiction.”)	[∴ q ∧ ~q]
∴ p , by proof-by-contradiction. QED	[∴ p]

Comments about Proof-by-Contradiction:

1.) An explicit statement that a contradiction has been reached is required:

Example: Assume that, earlier in the proof, it was pointed out that ½ is not an integer.

Assume that now it has just been established that ½ is an integer.

You can state: “Therefore, ½ is not an integer and ½ is an integer, which is a contradiction.”

Or, you can state: "Therefore, ½ is an integer, which contradicts the fact that ½ is not an integer."

Notice that this second wording combines the [∴ ~q] task and the [∴ q ∧ ~q] task into one sentence.

Thus, a proof-by-contradiction requires the use of one of the terms “contradicts” or “contradiction”.

2.) After arriving at a contradiction, you must immediately conclude the negation of the supposition.

3.) You are not allowed to start a proof-by-contradiction with the phrase “Suppose not.” Instead, you must “Suppose” the explicit wording of the negation of the statement to be proved.

4.) Use “Proof-by-Contradiction” to prove that something does not exist or that an object does not have a particular property.

Example:	Task
To Prove: There does not exist an integer which is both even and odd.	
Proof: (Proof-by-Contradiction)	
Suppose there exists an integer n which is both even and odd.	[Suppose ~p]
Note, ½ is not an integer.	[∴ q]
By definitions of “even” and “odd”, there exist integers k and ℓ such that n = 2k and n = 2ℓ + 1 .	
∴ 2k = 2ℓ + 1 , by substitution. ∴ 2 (k – ℓ) = 1 . ∴ (k – ℓ) = ½ .	
∴ ½ is an integer since (k – ℓ) is an integer.	[∴ ~q]
∴ ½ is an integer and ½ is not an integer, a contradiction	[∴ q ∧ ~q]
∴ There does not exist an integer which is both even and odd, by proof-by-contradiction. **QED**	[∴ p]

Proof-by-Contraposition:

The method of Proof-by-Contraposition to prove an "If A, Then B" statement involves the writing of a direct proof of its contrapositive statement: "If Not B, Then Not A."

The Design for Proofs of Conditional Statements by Contraposition

To Prove: IF p, THEN q .	Tasks for proving $p \rightarrow q$
Proof: (by Contraposition)	
Suppose ~q .	[Suppose ~q]
. . .	
. . .	
$\therefore$ ~p	[$\therefore$ ~p]
[$\therefore$ IF ~q, THEN ~p .] [Optional]	[$\therefore$ $\sim q \rightarrow \sim p$]
$\therefore$ IF p, THEN q , by Contraposition.	[$\therefore$ $p \rightarrow q$]

QED

Proposition 4.5.4: For all integers n, if n^2 is even, then n is even.

Proof: [by Contraposition]

Let n be any integer. [We will prove the statement: "If n is not even, then n^2 is not even."]

Suppose n is not even.

Then, by the Parity Corollary (and by Elimination), n is odd.

Therefore, there exists an integer k such that $n = 2k + 1$.

$\therefore$ $n^2 = (2k + 1)^2$, by substitution,

$= (4k^2 + 4k) + 1$

$= 4(k^2 + k) + 1 = 2(2(k^2 + k)) + 1$

$= 2t + 1$, where $t = 2(k^2 + k)$, which is an integer.

$\therefore$ n^2 is odd, by definition of "odd".

$\therefore$ n^2 is not even, since no integer is both even and odd;
[As was proved on the other side of this page]

Therefore, if n^2 is even, then n is even, by proof-by-contraposition.

$\therefore$ For all integers n, if n^2 is even, then n is even , by Direct Proof. QED